XXXII. *On the Preceſſion of the Equinoxes produced by the Sun's Attraction. By the Rev. Mr.* Iſaac Milner, *M. A. and Fellow of* Queen's College, Cambridge; *communicated by the Rev. Dr.* Shepherd, *F. R. S.*

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IF the actions of the Sun and Moon upon the different parts of the earth were equal; or if the earth itſelf were perfectly ſpherical, and of an uniform denſity from the center to the ſurface; in either caſe the attractions of thoſe remote bodies would have no effect on the poſition of the terreſtrial equator, and the equinoctial points would conſtantly be the ſame in the heavens. But it was impoſſible to give the earth a rotatory motion round an axis without giving at the ſame time a centrifugal force to its parts. This force is greateſt at the equator, and is in a contrary direction to that of gravity; on either ſide of the equator the force is leſs; and, moreover, only part of its effects is oppoſed to that of gravity. It is uſual in determining the figure of the earth to conſider the whole maſs as in a ſtate of fluidity, and the different columns as ſuſtaining one another at the center. If the earth be conſidered as a hard body, firmly cohering in its parts by ſome other force beſides that of gravity, it does not ſeem neceſſary that the different columns ſhould be ſuppoſed to ſuſtain each other at the center, though in both caſes the direction of gravity muſt at every point of the ſurface be perpendicular to the tangent of the figure. But we know, that there is a conſiderable quantity of water upon the ſurface of the earth; and, therefore, if the equatorial regions were not higher than the polar, they certainly would be overflowed by the Ocean, which is contrary to experience; and for this reaſon the proportion of the diameters of the earth, determined upon the falſe ſuppoſition of an entire fluidity, cannot differ much from the truth.

§ 2. But the preceſſion of the equinoxes, which depends upon the unequal actions of the Sun and Moon on the protuberant parts of the earth at the equator, will not be the ſame in theſe different hypotheſes; at leaſt we can never be certain that it will be ſo until we have computed their effects, and the computation itſelf muſt proceed on different principles. Suppoſe the earth to be fluid under the form of an oblate ſpheroid; or, what is more ſimple, ſuppoſe the region of the equator to be ſurrounded with a ring of fluid matter, and the unequal action of the Sun will diſturb the figure of the ring, and communicate a motion to its parts. Suppoſe we knew the preciſe diſturbing force of the Sun upon any one particle of this ring according to its ſituation; in that caſe we could eaſily find the velocity which would be communicated to ſuch a particle in any given time; but the mutual actions of the fluid particles upon each other could never be exactly eſtimated, much leſs their effects in endeavouring to turn the whole earth round its center. However, it is eaſy to ſee, that in the caſe of a hard ring of matter cohering cloſe with the ſurface of the earth at the equator, both the law by which the particles act on each other, and on the whole maſs of the earth, will be widely different from the caſe of fluidity, and the effects much greater in altering the poſition of the axis of rotation.

To explain this by an eaſy example (fig. 1,) let A, B, and C, repreſent three ſmall bodies in the ſame horizontal line AE. Suppoſe A to deſcend by any accelerating force as gravity; B to deſcend by the ſame force, a leſs or a greater; and C not to be acted upon at all: in every one of theſe caſes the bodies A and B will deſcend with their reſpective velocities, and the body C will preſerve its ſituation. If A and B are ſmall particles of fluid of any form, and C a hard one, and if the particle A be placed in contact with B, have its center of gravity a little above the center of gravity of B, and is acted upon by the greater accelerating force; in this caſe we may conceive how the action of A may diſturb the motion of B, and in the ſame way how the hard particle C may receive a ſmall motion from the actions of A and B. Then this motion muſt be extremely little compared with the whole motions of A or B, and ſtill a great deal leſs if C be ſtrongly connected with a ſtring of hard particles along the line CE, ſo that C cannot be moved without the whole line CE turning round the immoveable center E. Now if A, B, and C, be ſuppoſed hard particles firmly connected to the lever AE, then it is plain that the velocity of C, whatever it is, muſt be in proportion to that of A and B as their reſpective diſtances from E the center of motion, and this, whatever the impulſive forces are with which A and B are urged in their reſpective directions.

The body C being ſtill ſuppoſed void of gravity, let the bodies A and B be urged by forces perpendicular to AE in any ſmall equal times through the unequal ſpaces s and s', and let the magnitudes of the bodies be repreſented by A, B, and C reſpectively. Then the ſpace through which A is actually urged in that time will eaſily appear from mechanics to be repreſented by

$\frac{\overline{A \times AE \times s + B \times BE \times s \times AE}}{A \times AE^2 + B \times BE^2 + C \times CE^2}$, and the ſpace deſcribed by C is $\frac{\overline{A \times AE \times s + B \times BE \times s \times CE}}{A \times AE^2 + B \times BE^2 + C \times CE^2}$. See article the 13th.

§ 3. The preceding article being well underſtood, whatever doubts may remain concerning the motion of a ring of matter conſidered as detached from the earth, we may be certain that the motion of the nodes of the equator can never be the ſame, whether we ſuppoſe the ring at the equator to be fluid and to reſt upon the ſurface of the earth, partaking of the diurnal motion, or whether we ſuppoſe it hard and compact, and by its coheſion communicating a proportional degree of motion to the different parts of the earth. In fact, the problem of the preceſſion of the equinoxes, which has hitherto been conſidered as extremely difficult, and in its ſolution drawn out by authors to an immeaſurable length, requires no principles but the received doctrine of motion, and the application of the lever, which have been made uſe of in the laſt article. In that article we ſuppoſed the bodies A and B to be impelled by different forces in parallel lines, and we eſtimated the real ſpace, which either A or C in any ſmall time would deſcribe in conſequence of thoſe impulſive forces and their mutual connection by an inflexible lever. Now this is preciſely what is required to be done in the caſe of the Sun's unequal action on the protuberant parts of the equator. The exceſſes or defects of that unequal action are to be conſidered as forces applied to thoſe parts, which would move them according to the different circumſtances through unequal ſpaces proportional to the forces in equal times of action, provided the particles were at liberty to move freely in the directions in which they are urged; and, laſtly, the real ſpace muſt be computed through which a particle moves at ſome known diſtance from the center of the earth in conſequence of theſe various forces. This whole proceſs will not differ from the eaſy example already deſcribed, except in the length of the calculation, and the proper management of the doctrine of fluxions; and it ſeems adviſable in difficult ſubjects always to begin with ſimple inſtances before we proceed to thoſe which are more complex, and to diſtinguiſh the algebraical operations from the principles upon which they are founded.

§ 4. In order to determine how much any particle of the earth is affected by the unequal action of the Sun (fig. 2.), let CADB repreſent the earth, S the Sun at a great diſtance, and CD a plane perpendicular to the line ST joining the centers of the Sun and earth. If SK or ST repreſent the accelerating force of the Sun on a particle at the earth's center, and SL be taken to SK in the duplicate ratio of ST to SP; SL will repreſent the attraction on any particle P, and by the reſolution of motion T*m* or PL will repreſent the perturbing force of the Sun on the ſame particle. By the conſtruction $SL : SK :: SK^2 : SP^2$, and by diviſion $KL : SK :: SK^2 - SP^2 : SP^2 :: \overline{SK + SP} \times PK : SP^2$ and PL or T*m* is nearly equal to 3PK, and as 3PK is to SK or ST, ſo is the ſpace deſcribed by P in any ſmall time in the direction PK, to the ſpace deſcribed in the ſame time by the center of the earth in conſequence of the Sun's attraction. This laſt ſpace is equal to $\frac{\dot{z}^2}{2ST}$ where z repreſents the arc deſcribed by the earth's center during any ſmall motion in its orbit, and the former is equal to $\frac{3PK \times \dot{z}^2}{2ST^2}$. This is the ſpace which would be deſcribed by P in the direction PK if the particle was at liberty to move freely. Let us at preſent ſuppoſe that no other particle is diſturbed by the Sun's attraction except this one, and then proceed to enquire into the effects of this diſturbance when P by its coheſion communicates a motion to the different parts of the earth, which is farther conſtrained to turn round an axis T, the common interſection of the plane CD and the terreſtrial equator. From the laws by which motion is communicated, and the property of the lever, it eaſily appears, as in the ſecond article, that the ſpace through which any particle of the earth's equator is impelled at the greateſt diſtance from the axis T, is to $\frac{3\mathrm{PK}\dot{z}^2}{2\mathrm{ST}^2}$, the ſpace which would be deſcribed in the ſame time by any particle at liberty, the magnitude of which is repreſented by P, as P × KT × the radius of the equator to the ſum of all the particles of the earth multiplied into the ſquares of their reſpective diſtances from the ſaid axis.

To compute this ſum in the eaſieſt way, and by an approximation, which is quite ſufficient when the polar and equatorial diameters differ little from one another; let DPE (fig. 3.) be a ſphere whoſe radius is unity, divided into an infinite number of thin cylindrical ſurfaces, whoſe baſes are the circles NAQ; it is obvious, that all the particles in any one of theſe ſurfaces are at the ſame diſtance $\mathrm{CA} = x$ from the axis of motion perpendicular to the plane of the circle NAQ. Call AP, y, and A, the area of the circle DPE and the fluent of $4\mathrm{A}x^3\dot{x}y$, or of $-4\mathrm{A}x^2y^2\dot{y}$, becauſe $x\dot{x} = -y\dot{y}$ gives the ſum of all the particles in the ſphere multiplied into the ſquares of the reſpective diſtances from the axis. This fluent corrected is equal to $\frac{8\mathrm{A}}{15}$, and muſt now be diminiſhed in the ratio of 1 to $1-2p$, if we ſuppoſe the earth to be an oblate ſpheroid whoſe equatorial diameter is to the polar as 1 to $1-p$;

and, laſtly, the ſpace deſcribed by a particle at the greateſt diſtance from the axis is equal to $\frac{45\text{P}\times\text{PK}\times\text{KT}\times\dot{z}^2}{16\text{A}\times\text{ST}^2\times\overline{1-2p}}$.

§ 5. In fig. 4. let PIA*p*DK repreſent the earth orthographically projected on the plane of the ſolſtitial colure, P, *p*, the poles, IK a leſſer circle parallel to the equator, and P*ape* a ſphere deſcribed with the polar radius PT: then, ſince the particles without the globe only are concerned in changing the poſition of the axis of rotation, let L repreſent ſuch a particle ſituated in the circumference of the circle IK, and by the preceding article its effect will be $\frac{45\text{L}\times\text{LM}\times\text{MT}\times\dot{z}^2}{16\text{A}\times\text{ST}^2\times\overline{1-2p}}$, and by the ſame way of reaſoning, when two equal particles L, *l*, are ſuppoſed to be diſturbed by the Sun's attraction, the ſpace deſcribed by that point of the equator, which is at the greateſt diſtance from the axis of rotation or the common interſection of the plane CD and the equator A, will be equal to $\frac{45\dot{z}\times\overline{\text{L}\times\text{LM}\times\text{MT}+l\times lm\times m\text{T}}}{16\text{ST}^2\times\text{A}\times\overline{1-2p}}$, and the ſame argument holds for every other particle without the ſphere.

The ſum of all the L×LM×MT+&c. muſt now be found; and for this purpoſe Sir ISAAC NEWTON's conſtruction is, perhaps, as convenient as any that has hitherto appeared. In the ſame figure N*n* is parallel, and *xy* perpendicular, to CD; take L*x*=*xl*, and let *m*, *n*, repreſent the ſine and coſine of the angle CTP to the radius unity. It is eaſy to prove in his way that $\mathrm{L}\times \mathrm{LM}\times \mathrm{MT} + l\times lm\times m\mathrm{T}$ is equal to $2\mathrm{L}\times m\times n\times \overline{\mathrm{L}x^2 - \mathrm{T}x^2}$, and the fluent of $\mathrm{L}x^2$ multiplied into the fluxion of the circular arc $\mathrm{L}x$ is eaſily found in the following manner, without having recourſe to tables of fluents, or the methods of continuation.

From a known analogy the fluxion of the arc $\mathrm{L}x$ is to the fluxion of its verſed ſine as the radius $\mathrm{L}x$ of the ſame circle to $\mathrm{L}x$ the right ſine. $\mathrm{L}x$ multiplied into the fluxion of the verſed ſine is the fluxion of the area of the ſemicircle $\mathrm{L}l$, and calling $\mathrm{I}x$, y, the fluent of $\mathrm{L}x^2$ multiplied into the fluxion of the arc $\mathrm{L}x$ is evidently equal to $\frac{y^3\mathrm{A}}{2}$, where A ſtill repreſents the area of a circle whoſe radius is unity: $\mathrm{A}y$ is equal to the ſemi-circumference IK and $\mathrm{A}y\times \mathrm{T}x^2$ is equal to the fluent of $\mathrm{T}x^2$ multiplied into the ſame fluxion, and calling $\mathrm{T}x$, v, and ſubſtituting for $\mathrm{I}i$ its equal py, the ſum of all the $\mathrm{L}\times \mathrm{LM}\times \mathrm{MT} + \&c.$ in the annulus $\mathrm{I}i$ is equal to $mnp\mathrm{A}\times \overline{y^4 - 2y^2v^2}$. This laſt quantity multiplied into the fluxion of v, and the fluent taken by the common method when v is equal to TP or unity nearly, comes out $\frac{4p\mathrm{A}mn}{15}$ and twice this quantity gives the ſum of all the $\mathrm{L}\times \mathrm{LM}\times \mathrm{MT}$, without the whole ſphere $\mathrm{P}ap$, and therefore the ſpace deſcribed by a particle of the equator in the circle of the Sun's declination while the center of the earth is carried through the ſpace $\dot{z}$ of its orbit is equal to $\frac{3pmn\dot{z}^2}{2ST^2 \times \overline{1-2p}}$, and may be ſuppoſed to be equal to $\frac{3pm\dot{z}^2}{2ST}$, the alteration produced by the correction in art. 4. on account of the ſpheroidical figure of the earth being too inconſiderable to affect the concluſion.

§ 6. We are to obſerve, that the ſpace $\frac{3pmn\dot{z}^2}{2ST}$ deſcribed by that point of the equator, which is the interſection of the circle of the Sun's declination, is generated by the perpetual attraction of the Sun. This attraction may be reckoned conſtant during the very ſmall time of the earth's deſcribing $\dot{z}$ in its annual motion; and, therefore, the ſaid point of the equator, at the end of that time, will have acquired a velocity which would carry it through $\frac{3pmn\dot{z}^2}{ST^2}$ in the ſame time.

§ 7. Let T repreſent the time of the earth's revolution in its orbit, t the time of its rotation round its axis, and ſuppoſe $\dot{w}$ to be a ſmall arc ſimilar to $\dot{z}$ in a circle whoſe radius is unity. In figure 5. let AQ be the equator, and take Ab equal to $\frac{\dot{w}T}{t}$, and bt perpendicular to Ab equal to $3pmn\dot{w}^2$, and Ab, bt, will repreſent the directions and quantities of the two different motions of the point A, and conſequently At will be the direction of the new equator, and as Ab or At is to bt, ſo is the radius unity to the ſine of the angle tAb, and if AQ or AG be taken equal to a quadrant, GQ the meaſure of the angle GAQ is equal to $\frac{3pmnwt}{T}$.

§ 8. Suppoſe S the Sun's place in the ecliptic NS, N the equinoctial point, NA the Sun's right aſcenſion, and ro a perpendicular on AN: then ro is to GQ as the ſine of AN to the radius, and rN to ro as the radius to the ſine of the angle at N the inclination of the ecliptic to the equator, and, *ex æquo perturbatè*, rN to GQ as the ſine of AN to the ſine of N and rN the ſmall preceſſion of the equinoxes is equal to $3pmn\,\frac{w \times t}{T} \times \frac{\text{ſin. NA}}{\text{ſin. N}}$.

§ 9. In the ſpherical triangle ASN, the ſine of AN = cotang. N × tang. AS $= \frac{m \times \text{coſ. N}}{n \times \text{ſin. N}}$, and farther the ſine of SN $= \frac{m}{\text{ſin. N}}$ and rN is equal to $\frac{3pt \times \overline{\text{ſin. } w}|^2 \times w \times \text{ coſ. N}}{T}$, whoſe fluent or $\frac{3pt}{4T} \times \overline{2w - \text{ſin. } 2w} \times$ coſ. N gives the preceſſion of the equinoxes during the Sun's motion through the arc NS of the ecliptic: when NS is equal to a circle, then the whole fluent becomes equal to $\frac{3ptA}{T} \times$ coſ. N, and as $2T$ is to $3pt \times$ coſ. N, ſo is $60 \times 60 \times 360$ to $21'' + 6'''$ the annual preceſſion of the equinoxes in ſeconds produced by the Sun's attraction.

§ 10. We might now proceed in a ſimilar way to inveſtigate the effects of the Moon's diſturbing force, the rotation of the earth's axis, and the equation of the preceſſion; but ſince theſe propoſitions are purely mathematical, and the computations have already been gone through by other authors, it will be needleſs to repeat them here.

§ 11. NEWTON was the firſt who attempted to explain the preceſſion of the equinoxes from its cauſes. Since his time various other ſolutions have been given us by the moſt celebrated mathematicians; and it deſerves to be noticed, that, in a caſe where there can be little doubt that he was miſtaken, other authors have found it difficult to agree among themſelves in differing from him. M. D'ALEMBERT, in the year 1749, printed a treatiſe expreſsly on the ſubject, and has ſince ſaid (a), that himſelf is acknowledged to be the firſt who determined rightly the method of ſolving ſuch problems. EULER, DE LA GRANGE, FRISIUS, SILVABELLE, WALMESLEY, SIMPSON, EMERSON, have each conſidered the ſubject, and perhaps the importance of the enquiry would juſtify a minute examination into the cauſe of the agreement or diſagreement of their ſeveral methods; but I am deterred from entering into ſuch a diſcuſſion by the length of time which it would require; eſpecially as I think thoſe who have read the authors mentioned will eaſily conceive the ſubſtance of what I ſhould have to obſerve, and to thoſe who have not read them I ſhould hardly be able to ſay any thing intelligible.

(a) D'ailleurs, des géometres, vraiment capables d'apprècier mon travail, ont abondamment ſuppléé à tout autre temoignage, en declarant que j'ai ouvert le premier la route pour reſoudre ce genre de queſtions. See Opuſc. Math. vol. V. ſur la Préceſſion des Equinoxes.

§ 12. The above ſolution, if it had no other advantages, is, I apprehend, much more conciſe than any that has hitherto been given. Abſtracted from what is ſaid by way of illuſtration, articles 4th to 9th contain all the calculation requiſite, and as I have ſtudiouſly avoided the ambiguous uſe of the terms *force*, *vis*, *efficacia*, *momentum*, &c. as well as every doubtful repreſentation of times, ſpaces, and velocities, which are often ſubſtituted by authors in equations, I believe the whole proceſs will appear eaſy, and the evidence upon which the concluſion reſts be exactly aſcertained.

§ 13. The principles deſcribed in articles 2. and 4. depend upon the third law of motion, and the property of the lever, and are demonſtrated in the following manner. Every thing remaining the ſame as in art. 2. (fig. 6.) let AV and BR, perpendicular to the right line or axis AE, repreſent the forces and directions with which thoſe bodies are reſpectively urged, when at liberty to move

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freely in thoſe directions; and let Av, Br, Cc, repreſent the accelerative forces of the reſpective bodies, as altered by their mutual actions upon each other: then, becauſe $C \times Cc$ is the moving force gained by C, and $A \times vV + B \times rR$ the moving force loſt by A and B, regard being had to the lengths of the different levers AE, BE, we ſhall have $A \times vV \times AE + B \times rR \times BE$ equal to $C \times Cc \times CE$, that is, $A \times AE \times \overline{AV - Av} + B \times BE \times \overline{BR - Br}$ equal to $C \times Cc \times CE$, and by tranſpoſition $A \times AE \times AV + B \times BE \times BR$ equal to $C \times CE \times Cc + A \times AE \times Ar + B \times BE \times Br$. Let s, $\acute{s}$, repreſent, as in art. 2. the ſpaces which would be deſcribed by the bodies A and B at liberty in any very ſmall portion of time, and let x be the ſpace which A actually deſcribes in that time when connected with B and C by the lever AE. The quantities $\frac{x \times BE}{AE}$, $\frac{x \times CE}{AE}$, will then be the ſpaces deſcribed by B and C reſpectively; and, laſtly, becauſe the ſpaces deſcribed in given times are as the accelerating forces, the above equation gives x equal to $\frac{A \times AE \times s + B \times BE \times s \times AE}{A \times AE^2 + B \times BE^2 + C \times CE^2}$.

The ſame method extends itſelf eaſily to more difficult caſes, and by its aſſiſtance ſeveral very important theorems are briefly demonſtrated.

§ 14. The reaſoning made uſe of in art. 6. will appear very evident to any one moderately verſed in the elements of mechanics and the doctrine of moving forces; and therefore I muſt believe that it is by miſtake that one author of note entirely omits ſo neceſſary a ſtep which affects the concluſion by juſt one half. When a body moves with any velocity in the direction AM (fig. 7.) which would carry it through the ſpace AD in a ſmall particle of time, and any force which may be reckoned conſtant for that time urges the body through the ſpace DC perpendicular to AM, the body at the end of that time will arrive at the point C; but joining AC we are not to ſuppoſe, that, if that force ceaſed to act, the body would proceed in the direction ACL: for take C*m* equal and parallel to AD, and C*d* in CD produced equal to 2CD, and the direction of C at that point will be C*l*, the diagonal of the parallelogram C*dlm*.

Thus when a body revolves in any curve by a centripetal force (fig. 8.) we may, with Sir ISAAC NEWTON, ſuppoſe the curve to be compoſed of an indefinite number of right lines, and the body to move either in the chords or the tangents of the curve; but then we are to take care that we make not ſuppoſitions inconſiſtent with each other. Let the curve be a circle, and AD a tangent at the point A the direction of the body's motion when it arrives at that point, and let DC, parallel to the diameter AL be the effect of the centripetal force: then, if we ſuppoſe the body to move along the chord AC, and ſay, that the angle CAD meaſures the deflection of the path in the time of the body's moving through the arc or chord AC, we ſhall miſtake by one half of the true quantity; for draw the tangent at C, and ſince A*d* is equal to *d*C from the property of the circle, the angle C*d*D of deviation is equal to twice the angle CA*d*. The practice of NEWTON in a ſimilar caſe, where he is inveſtigating the horary motion of the lunar nodes in a circular orbit, is entirely conſiſtent with this. See the Principia, lib. III. prop. 30.

§ 15. M. D'ALEMBERT has lately charged SIMPSON's account of the preceſſion of the equinoxes with ſome miſtakes of this nature in his ſecond lemma; but, in juſtice to SIMPSON, I muſt ſay, that, whatever other defects there may be in his paper, I am convinced, after the moſt diligent attention, that thoſe alluded to are without foundation.

§ 16. Sir ISAAC NEWTON firſt obſerved, that an homogeneous globe could not poſſibly retain many diſtinct motions, without compounding them all into one, and revolving with a ſimple and uniform motion about an invariable axis. When two forces impreſs upon a globe two diſtinct circular motions(*b*), he briefly concludes in

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(*b*) See Principia, lib. I. prop. LXVI. coroll. 22.

his way, from the laws of motion, that it is the ſame thing as if thoſe two forces were at once impreſſed in the common interſection of the equators of thoſe motions, and upon this principle we ſuppoſed A*t* in art. 7. to be the direction of the new equator. In order to remove any doubts that might ariſe about the juſtneſs of this mode of compounding motion, FRISIUS has given a geometrical demonſtration of the principle: but the thing may be ſhewn much more eaſily in the following manner. Suppoſe (fig. 9.) RB, AB, to be two axes about which every point in the plane ABPR tends to move with velocities as the reſpective diſtances from the axes; let PQ perpendicular to AB be to PR perpendicular to RB as the angular velocity of P about RB to the angular velocity of the ſame point about AB, and let the velocities be in contrary directions: then, I ſay, every point in the plane will move with a velocity proportional to its diſtance from the axis PB. Firſt, it is evident, that any point C in the axis RB will move round PB with a velocity proportional to its diſtance C*m*: for the point C lying in the axis RB has no velocity round RB, and CM is proportional to CN. Draw P*c* parallel to AB, and any point D in that line will move with a velocity proportional to DV, which is perpendicular to PB, for the following reaſon.

The velocity of D is equal to the difference of its two velocities round the reſpective axes RB, AB, or the difference of D's velocity round RB, and P's velocity round AB, ſince all the points in PC move with the ſame velocity round AB. Draw DT parallel to CR, and this difference will be proportional to PT, becauſe the velocities of P round RB, AB, are ſuppoſed equal to each other, and PT is proportional to DV, and every point in the plane moves round PB with a velocity proportional to its diſtance; and the ſame thing may be ſhewn when any point is taken without the plane ABCPR.

§ 17. Becauſe any point C in the axis RB moves with the ſame velocity round PB as it does round AB, the angular velocities round the two axes AB, PB, will be to each other inverſely as their reſpective diſtances CN, C*m*; and becauſe CN : C*m* :: PB : PC and PR : PQ :: PC : CB, it follows, *that* PB *the diagonal of the parallelogram* PGBC *will repreſent the angular velocity of the revolving plane, when* BG, BC, *are taken to each other as the angular velocities of the ſame plane round thoſe reſpective axes.*

§ 18. From this it clearly follows, that the reaſon given by SIMPSON (*c*), in his miſcellaneous tracts, of the difference between his own ſolution and that of NEWTON in the Principia, cannot poſſibly be the true one. "It

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(*c*) Pages 44. and 45.

" appears further," ſays he, " by peruſing his thirty-ninth " propoſition, that he there aſſumes it as a principle, that " if a ring encompaſſing the earth at its equator, but de- " tached therefrom, was to tend or begin to move about " its diameter with the ſame accelerative force or angular " celerity as that whereby the earth itſelf tends to move " about the ſame diameter through the action of the " Sun, that then the motion of the nodes of the ring and " of the equator would be exactly the ſame."

The principle is certainly implied in NEWTON's proof, and is capable of the moſt rigid demonſtration, art. 16, 17.

§ 19. It will be aſked then, where is the fault of NEWTON's reaſoning? How comes his concluſion to be too little by above one half? It is acknowledged on all hands that there is an error in his third lemma; but then the correction of that error makes only a very ſmall alteration in the reſult.

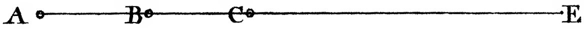
It is impoſſible for any one to form a complete judgement of his method without going through the whole of his calculations, which pre-ſuppoſes that the mean motion of the lunar nodes is computed. This motion may be conciſely determined and exactly enough for the purpoſe from prop. 30. of the Principia, and from thence is inferred the motion of the nodes of a ſatellite revolving in the plane of the equator at the ſurface of the earth with a velocity equal to that of the earth round its axis. We are then to ſuppoſe, that the *mean* motion of the nodes of a ſatellite, revolving with ſuch a velocity, is the ſame with the motion of the nodes of a ring of rigid matter ſurrounding the earth at its equator, and revolving with the ſame velocity. This laſt hypotheſis is admitted by SIMPSON, who thinks that his own ſecond lemma contains a full demonſtration of the point. For my own part, I believe with FRISIUS, that we are to look here for the material error in NEWTON's ſolution of the problem. It is evident, that the *true* motion of the nodes of the ſatellite, and the ring of matter, are not the ſame; and it is by no means obvious, that their mean motions are ſo. The mean motion of the nodes of a ring of hard matter cohering together is very eaſily computed by the method in art. 4th to the 9th, and turns out nearly double the mean motion of a Moon revolving at the ſurface of the earth with the ſame velocity.

It is a very intereſting enquiry to find out the real cauſe of the miſtake in the Principia, lib. III. prop. 39.; and therefore at a future opportunity I may, perhaps, conſider this particular part of the ſubject more attentively. I have long been ſatisfied with the account already given, and ſhould probably have remained ſo, if M. D'ALEMBERT (d), in his Opuſc. vol. V. had not perſiſted to affirm, that the mean motion of the nodes of the ring of matter and of the ſatellite were the ſame.

This opinion of ſo celebrated a mathematician raiſes ſcruples in one's mind; and ſhews, that when we venture to differ from Sir ISAAC NEWTON in theſe matters, it is with the utmoſt difficulty that we can arrive at certainty.

(d) Il n'y a de parité que dans le mouvement *moyen* de ces deux anneaux, ou de l'anneau ſolide et de la lune; les mouvemens *inſtantanés* ſont très differens de part et d'autre; ainſi la comparaiſon du mouvement de l'anneau avec celui de la lune, ſerviroit tout au plus à trouver le mouvement *moyen* de l'anneau, ou de la préceſſion des équinoxès, mais nullement à déterminer la nutation de l'axe et l'équation de la préceſſion.

Fig. 1.



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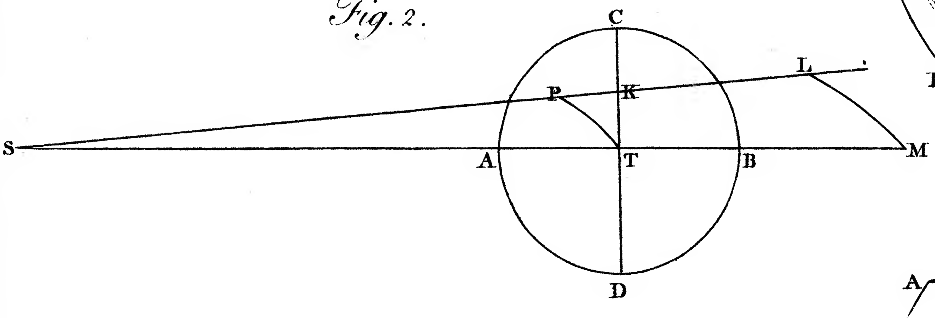

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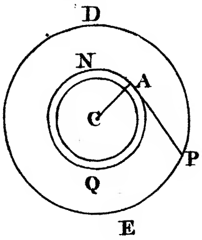

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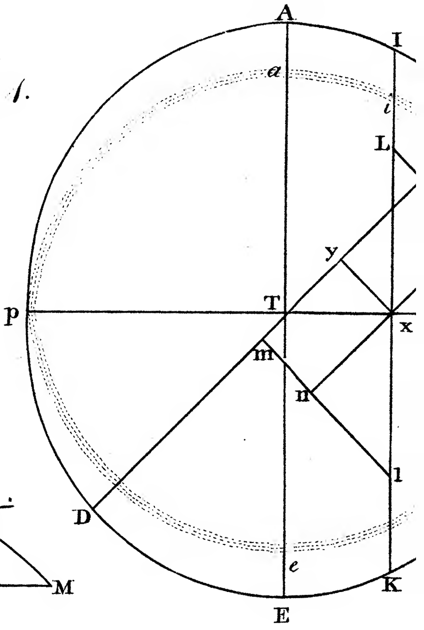

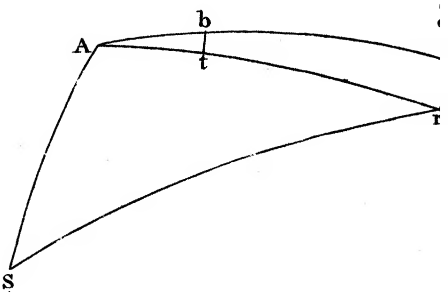

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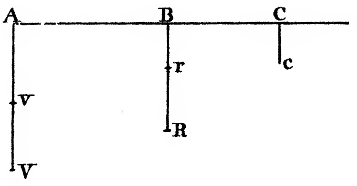
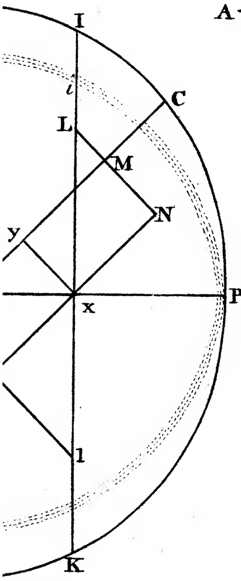




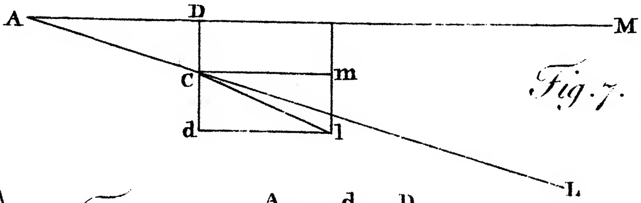
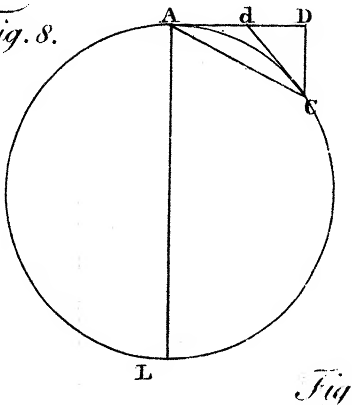


Fig. 7.

Fig. 8.



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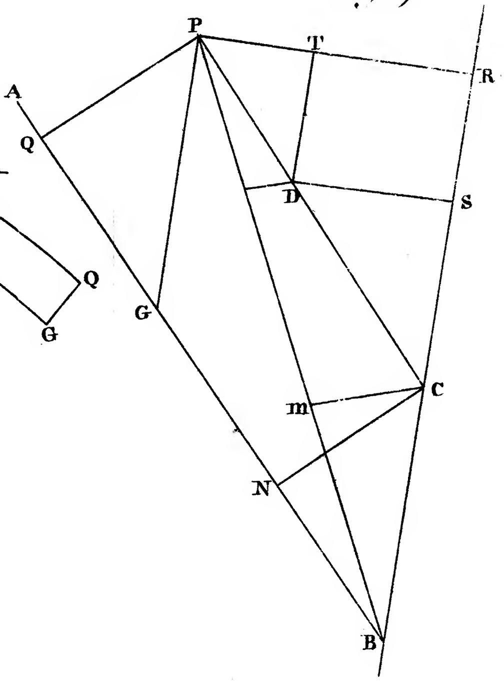

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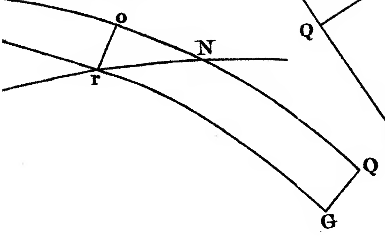

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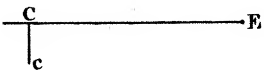


Fig 1

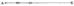


Fig 2

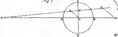
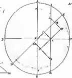


Fig 3

Fig 4



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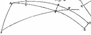

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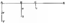

Fig 7

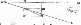

Fig 8

Fig 9